GAUSS-ELIMINATION METHOD

This is an elimination method and it reduces the given system of equation to an equivalent upper triangular system which can be solved by back substitution method.

Principle : $[A/B] \longrightarrow [U/K]$

GAUSS-JORDAN METHOD

This method is a modification of Gauss-elimination method. The coefficient matrix A of the system $AX=B$ is reduced into a diagonal or a unit matrix and the solution is obtained directly without back substitution process.

PRINCIPLE : $[A/B] \longrightarrow [I/K]$

Compare Gauss elimination method and Gauss-Jordan method.

	Gauss-elimination method	Gauss-Jordan method
1.	Coefficient matrix is transformed into upper triangular matrix.	coefficient matrix is transformed into diagonal matrix.
2.	Direct method	Direct method
3.	We obtain the solutions by back substitution method.	No need of back substitution method.

Problems :

1. Solve the system of equations by Gauss-elimination method.

$$10x - 2y + 3z = 23$$
$$2x + 10y - 5z = -33$$
$$3x - 4y + 10z = 41$$

Solution :

The given system is equivalent to

$$\begin{bmatrix} 10 & -2 & 3 \\ 2 & 10 & -5 \\ 3 & -4 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 23 \\ -33 \\ 41 \end{bmatrix}$$

$$AX = B$$

Here $[A, B] = \begin{bmatrix} 10 & -2 & 3 & 23 \\ 2 & 10 & -5 & -33 \\ 3 & -4 & 10 & 41 \end{bmatrix}$

Now, we will make the matrix A as a upper triangular.

Fix the first row, change 2 and 3 row with row 1.

$$[A, B] \sim \begin{bmatrix} 10 & -2 & 3 & 23 \\ 0 & 52 & -28 & -188 \\ 0 & -34 & 91 & 341 \end{bmatrix} \begin{matrix} \\ R_2 \leftrightarrow 5R_2 - R_1 \\ R_3 \leftrightarrow 10R_3 - 3R_1 \end{matrix}$$

Fix 1 and 2 row, change 3 row with 2nd row.

$$\sim \begin{bmatrix} 10 & -2 & 3 & 23 \\ 0 & 52 & -28 & -188 \\ 0 & 0 & 3780 & 11340 \end{bmatrix} \longrightarrow (1)$$

$R_3 \leftrightarrow 52R_3 + 34R_2$

This is an upper triangular matrix.

From (1) we get (by back substitution)

$$3780z = 11340$$

$$z = \frac{11340}{3780} = 3.$$

$$52y - 28z = -188$$

$$52y - 28(3) = -188$$

$$52y - 84 = -188$$

$$52y = -188 + 84 = -104$$

$$y = \frac{-104}{52} = -2.$$

$$10x - 2y + 3z = 23$$

$$10x - 2(-2) + 3(3) = 23$$

$$10x + 4 + 9 = 23$$

$$10x = 23 - 13 = 10$$

$$x = \frac{10}{10} = 1.$$

Hence the solution is $x = 1$, $y = -2$, $z = 3$

2. Solve the equation $x+y+z=9$, $2x-3y+4z=13$, $3x+4y+5z=40$ by Gauss- elimination method.

Solution :

$$(A/B) = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 2 & -3 & 4 & 13 \\ 3 & 4 & 5 & 40 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & -5 & 2 & -5 \\ 0 & 1 & 2 & 13 \end{array}\right] \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & 1 & 2 & 13 \\ 0 & -5 & 2 & -5 \end{array}\right] R_2 \leftrightarrow R_3$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & 1 & 2 & 13 \\ 0 & 0 & 12 & 60 \end{array}\right] R_3 \to R_3 + 5R_2$$

By back Substitution

$$12z = 60$$

$$z = \frac{60}{12} = 5$$

$$y+2z = 13$$

$$y + 2(5) = 13$$

$$y+10 = 13 \Rightarrow y = 13-10 = 3$$

$x+y+z=9$

$x+3+5=9$

$x=9-8=1.$

$\therefore$ The solution is $\boxed{x=1,\ y=3,\ z=5}$.

③ Solve the following system by Gauss-elimination method.

$5x_1+x_2+x_3+x_4=4$

$x_1+7x_2+x_3+x_4=12$

$x_1+x_2+6x_3+x_4=-5$

$x_1+x_2+x_3+4x_4=-6.$

Solution :

$$(A|B)=\left[\begin{array}{cccc|c} 1 & 1 & 1 & 4 & -6 \\ 1 & 7 & 1 & 1 & 12 \\ 1 & 1 & 6 & 1 & -5 \\ 5 & 1 & 1 & 1 & 4 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & 4 & -6 \\ 0 & 6 & 0 & -3 & 18 \\ 0 & 0 & 5 & -3 & 1 \\ 0 & -4 & -4 & -19 & 34 \end{array}\right] \begin{array}{l} \\ R_2 \to R_2-R_1 \\ R_3 \to R_3-R_1 \\ R_4 \to R_4-5R_1 \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & 4 & -6 \\ 0 & 6 & 0 & -3 & 18 \\ 0 & 0 & 5 & -3 & 1 \\ 0 & 0 & -4 & -21 & 46 \end{array}\right] \begin{array}{l} \\ \\ \\ R_4 \to R_4+\left(\frac{2}{3}\right)R_2 \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & 4 & -6 \\ 0 & 6 & 0 & -3 & 18 \\ 0 & 0 & 5 & -3 & 1 \\ 0 & 0 & 0 & -\frac{117}{5} & 234/5 \end{array}\right] R_4 \to R_4 + \left(\frac{4}{5}\right) R_3$$

By back substitution method

$$-\frac{117}{\cancel{5}} x_4 = \frac{234}{\cancel{5}}$$

$$-117 x_4 = 234$$

$$x_4 = \frac{234}{-117} = -2$$

$$5x_3 - 3x_4 = 1$$

$$5x_3 - 3(-2) = 1$$

$$5x_3 + 6 = 1$$

$$5x_3 = 1 - 6 = -5$$

$$x_3 = -5/5 = -1$$

$$6x_2 - 3x_4 = 18$$

$$6x_2 - 3(-2) = 18$$

$$6x_2 + 6 = 18$$

$$6x_2 = 18 - 6 = 12$$

$$x_2 = \frac{12}{6} = 2$$

$$x_1 + x_2 + x_3 + 4x_4 = -6$$

$$x_1 + 2 - 1 + 4(-2) = -6$$

$$x_1 + 1 - 8 = -6$$

$$x_1 - 7 = -6$$

$$x_1 = -6 + 7 = 1$$

$\boxed{\therefore \text{ The solution is } x_1 = 1,\ x_2 = 2,\ x_3 = -1,\ x_4 = -2}$

Homework problems :

1. Solve the system by Gauss-elimination method.

$$5x - y - 2z = 142$$
$$x - 3y - z = -30$$
$$2x - y - 3z = 5.$$

Ans: $x = 39.345$
$y = 16.793$
$z = 18.966$

2. Solve the equations by Gauss-elimination method.

$$2x + y + 4z = 12$$
$$8x - 3y + 2z = 20$$
$$4x + 11y - z = 33.$$

Ans: $x = 3$
$y = 2$
$z = 1.$

3. Solve by Gauss-elimination method.

$$3x + 4y + 5z = 18$$
$$2x - y + 8z = 13$$
$$5x - 2y + 7z = 20$$

Ans:
$x = 3$
$y = 1$
$z = 1$